Solutions for Practice for Final Exam 20100 Calculus I, CCNY, Spring 2003

Yegor Bryukhov

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1.

$$\lim_{x\to-1}\frac{(x+1)^{\frac{3}{2}}}{\sin(x^2-1)}=\lim_{x\to-1}\frac{x^2-1}{\sin(x^2-1)}\cdot\frac{(x+1)^{\frac{3}{2}}}{x^2-1}=\lim_{x\to-1}\frac{x^2-1}{\sin(x^2-1)}\cdot\lim_{x\to-1}\frac{(x+1)^{\frac{3}{2}}}{x^2-1}=$$

$$1\cdot\lim_{x\to-1}\frac{(x+1)^{\frac{3}{2}}}{x^2-1}=\lim_{x\to-1}\frac{(x+1)(x+1)^{\frac{1}{2}}}{(x-1)(x+1)}=\lim_{x\to-1}\frac{(x+1)^{\frac{1}{2}}}{(x-1)}=\frac{\lim_{x\to-1}(x+1)^{\frac{1}{2}}}{\lim_{x\to-1}(x-1)}=\frac{0}{-2}=0$$

2.

$$f'(x)=\left(5\sqrt{6-x}+\frac{2}{\sqrt[3]{2x^2}}\right)'=5\frac{1}{2\sqrt{6-x}}(-x)'+\left(\frac{2}{\sqrt[3]{2}}x^{-\frac{2}{3}}\right)'=$$

$$\frac{-5}{2\sqrt{6-x}}-\frac{2}{\sqrt[3]{2}}\cdot\frac{2}{3}x^{-\frac{2}{3}-1}=-\frac{5}{2\sqrt{6-x}}-\frac{2\sqrt[3]{4}}{3}x^{-\frac{5}{3}}$$

$$f'(2)=\frac{-5}{2\sqrt{6-2}}-\frac{8}{3\sqrt[3]{2\cdot2}}=\frac{-5}{2\sqrt{4}}-\frac{8}{3\sqrt[3]{4}}=-\frac{5}{4}-\frac{4\sqrt[3]{2}}{3}$$

3. (a)

$$y'=\left(\frac{3x-1}{x^3+x}\right)=\frac{(3x-1)'(x^3+x)-(3x-1)(x^3+x)'}{(x^3+x)^2}=$$

$$\frac{3(x^3+x)-(3x-1)(3x^2+1)}{(x^3+x)^2}=\frac{3x^3+3x-9x^3+3x^2-3x+1}{(x^3+x)^2}=\frac{-6x^3+3x^2+1}{(x^3+x)^2}$$

(b)

$$y'=\left((3x+1)\tan^2\frac{x}{2}\right)'=(3x+1)'\tan^2\frac{x}{2}+(3x+1)\left(\tan^2\frac{x}{2}\right)'=$$

$$3\tan^2\frac{x}{2}+(3x+1)2\tan\frac{x}{2}\left(\tan\frac{x}{2}\right)'=3\tan^2\frac{x}{2}+2(3x+1)\tan\frac{x}{2}\cdot\frac{1}{\cos^2\frac{x}{2}}\left(\frac{x}{2}\right)'=$$

$$3\tan^2\frac{x}{2}+2(3x+1)\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}=3\tan^2\frac{x}{2}+(3x+1)\frac{\sin\frac{x}{2}}{\cos^3\frac{x}{2}}$$

4. Volume of the ball $V = \frac{4}{3}\pi r^3$, inflation rate is $\frac{dV}{dr} = 1in^3/sec$, find the rate of change of diameter (as a function of time) when it's 2 ft ($D = 24in$, $r = 12in$). $\frac{dV}{dr} = \frac{4}{3} \cdot 3\pi r^2 \frac{dr}{dt} = 1$, so $\frac{dV}{dr}\Big|_{r=12} = 4\pi(12)^2 \frac{dr}{dt}\Big|_{r=12} = 1$ and $\frac{dr}{dt}\Big|_{r=12} = \frac{1}{4\pi(12)^2} = \frac{1}{4\cdot 144\pi}$. $D = 2r$, $\frac{dD}{dt} = 2\frac{dr}{dt}$, $\frac{dD}{dt}\Big|_{r=12} = 2\frac{1}{4\cdot 144\pi} = \frac{1}{2\cdot 144\pi} = \frac{1}{288\pi}$.

5. (a) In order for this function to be continuous, $-ax^3$ and x should have the same value at 0, and x and $b + 4\sqrt{x}$ should have the same value at 4. Both $-ax^3$ and x are 0 when x is 0 so first condition holds. According to the second condition $4 = b+4\sqrt{4} = b+8$ hence $b = -4$. So function is continuous when $b = 4$ and any a.

   (b) $\left(-ax^3\right)' = -3ax^2$ for $x = 0$, $-3ax^2 = 0$. But $x' = 1$, so from different directions we have different values of derivative at the same point, hence derivative does not exist at this point.

   (c) The question is not correct. $(-4 + 4\sqrt{x})' = 4\frac{1}{2\sqrt{x}} = \frac{2}{\sqrt{x}}$ for $x = 4$, $\frac{2}{\sqrt{x}} = 1$. But $x' = 1$, so from different directions we have same values of derivative at the same point, hence derivative does exist at this point.

6. (a)

$$m = \frac{f(a) - f(b)}{a - b} = \frac{2^2 - 3^2}{2 - 3} = \frac{4 - 9}{-1} = \frac{-5}{-1} = 5$$

   (b)

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{x^3 + 3x(\Delta x)^2 + 3x^2\Delta x + \Delta x^3 - x^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + \Delta x^3}{\Delta x} =$$

$$\lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + \Delta x^2 = \lim_{\Delta x \to 0} 3x^2 + \lim_{\Delta x \to 0} 3x\Delta x + \lim_{\Delta x \to 0} \Delta x^2 = 3x^2 + 0 + 0 = 3x^2$$

   (c)

$$dy = f'(x)dx = 3x^2 dx$$

   It's easy to find 2^3, so let's approximate 2.2^3 using value of the function and its differential at point 2.

$$f(2.2) = f(2) + dy|_{x=2,\, dx=0.2} = 2^3 + 3\cdot 2^2\cdot 0.2 = 8 + 3\cdot 4\cdot 0.2 = 8 + 2.4 = 10.4$$

7. (a)

$$\left(\frac{\sin y}{x^2} + y \sin \pi x = 1\right)'$$

$$\frac{y' \cos y \cdot x^2 - 2x \sin y}{x^4} + y' \sin \pi x + y\pi \cos \pi x = 0$$

$$y'\left(\frac{\cos y}{x^2}+\sin\pi x\right)=\frac{2\sin y}{x^3}-\pi y\cos\pi x$$

$$y'(x\cos y+x^3\sin\pi x)=2\sin y-\pi x^3 y\cos\pi x$$

$$y'=\frac{2\sin y-\pi x^3 y\cos\pi x}{x\cos y+x^3\sin\pi x}$$

(b) Let's substitute $x = 1$ into the original equation:

$$\sin y + y\sin\pi = 1$$

$\sin\pi = 0$, so we receive that

$$\sin y = 1$$

which holds when $y = \pi k$ for $k \in \mathbf{Z}$, in this case $\cos y = \cos\pi k = (-1)^k$
No let's return to y':

$$y'=\frac{2\sin y-\pi x^3 y\cos\pi x}{x\cos y+x^3\sin\pi x}=\frac{2-\pi y\cos\pi}{\cos y+\sin\pi}=$$

$$\frac{2+\pi y}{\cos y}=\frac{2+\pi\pi k}{(-1)^k}=(-1)^k(2+\pi^2 k)$$

(c)

$$y(0.95)=y(1)+dy|_{x=1,\ dx=-0.05}=\pi k+y'(1)dx=\pi k+((-1)^k(2+\pi^2 k))(-0.05)=$$

$$\pi k-0.05(-1)^k(2+\pi^2 k)$$

8. As I said in the class one more condition is missing and let's say it is $w = d$, i.e. width of the granary is equal to the "depth" of the granary (distance between front side and the wall). Let's use h for the height of the front side of the granary then $2h$ is the height of the granary at the wall. The front and top of the granary are rectangulars and two other sides are equal trapezoids. Parallel sides of these trapezoids are of length h and $2h$, distance between these sides is d (distance between front side and the wall). So the area of one of these trapezoids is $\frac{(h+2h)d}{2} = \frac{3dh}{2}$. The area of the front side is wh. One side of the top of the granary is w. Another side of the top is the hypotenuse of right triangle with two other sides d and $2h - h = h$, hence it's $\sqrt{d^2+h^2}$. So the area of the top is $w\sqrt{d^2+h^2}$. And the total area (top, front and two sides) is:

$$A=\frac{3dh}{2}+wh+w\sqrt{d^2+h^2}=\frac{3dh}{2}+dh+d\sqrt{d^2+h^2}=\frac{5dh}{2}+d\sqrt{d^2+h^2}$$

The volume of the granary is $V = w \cdot \frac{3dh}{2}$ - area of the side by granary width. You can think about the granary as rectangular box of height h and half of such box (diagonally cut) above it. The volume of the box is dhw and it's half - $\frac{dhw}{2}$, $\frac{3dhw}{2}$ totally. Having in mind that $w = d$ we get:

$$V = \frac{3d^2h}{2}$$

and according to the problem conditions $V = 10$, hence

$$\frac{3d^2h}{2} = 10$$

$$3d^2h = 20$$

$$h = \frac{20}{3d^2} \text{ or } d = \sqrt{\frac{20}{3h}}$$

we substitute this formula for h into the area:

$$A = \frac{5dh}{2} + d\sqrt{d^2 + h^2} = \frac{50}{3d} + d\sqrt{d^2 + \frac{400}{9d^4}} = \frac{50}{3d} + \sqrt{d^4 + \frac{400}{9d^2}}$$

or

$$A = \frac{5dh}{2} + d\sqrt{d^2 + h^2} = \frac{5h}{2}\sqrt{\frac{20}{3h}} + \sqrt{\frac{20}{3h}}\sqrt{\frac{20}{3h} + h^2} = 5\sqrt{\frac{5h}{3}} + 2\sqrt{\frac{100}{9h^2} + \frac{5h}{3}} =$$

$$5\sqrt{\frac{5h}{3}} + 2\frac{\sqrt{100 + 15h^3}}{3h}$$

As we agreed in the class, because of the complexity of the formula you could stop after you got formula for A with only one varialbe (w or d or h). Of course on Final you'll have to complete your solution.

9. **Domain** $x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4} > 0$ for all x, hence division by zero never happens and function defined for all x.

Intercepts y-intercept - $f(0) = \frac{1}{1} = 1$

x-intercepts:

$$f(x) = 0$$

$$\frac{x+1}{x^2 + x + 1} = 0$$

$$x + 1 = 0$$

$$x = -1$$

The function intersects x-axis at $x = -1$.

Symmetry Points of intersection with x-axis is not symmetrical (only -1, no 1), hence function is neither odd, nor even.

Assymptotes There is no division by zero, hence there is no vertical asymptotes. Horizontal asymptotes:

$$\lim_{x\to+\infty} \frac{x+1}{x^2+x+1} = \lim_{x\to+\infty} \frac{1+\frac{1}{x}}{x+1+\frac{1}{x}} = \left[\frac{1+0}{+\infty+1+0}\right] = 0$$

$$\lim_{x\to-\infty} \frac{x+1}{x^2+x+1} = \lim_{x\to+\infty} \frac{1+\frac{1}{x}}{x+1+\frac{1}{x}} = \left[\frac{1+0}{-\infty+1+0}\right] = 0$$

Both at $+\infty$ and $-\infty$ the function has the same horizontal asymptote $y = 0$.

Properties of the first derivative

$$f'(x) = \frac{1\cdot(x^2+x+1)-(x+1)(2x+1)}{(x^2+x+1)^2} =$$

$$\frac{x^2+x+1-2x^2-2x-x-1}{(x^2+x+1)^2} = \frac{-x^2-2x}{(x^2+x+1)^2} = \frac{-x(x+2)}{(x^2+x+1)^2}$$

it always exists (no division by zero), so the only case for critical points is when $f'(x) = 0$:

$$\frac{-x(x+2)}{(x^2+x+1)^2} = 0$$

$$x(x+2) = 0$$

Critical points are $x = 0$ and $x = -2$. Denominator is always positive, hence sign of the derivative is the same as sign of $-x(x+2)$:

	x	$x+2$	$x(x+2)$	$-x(x+2)$	$f'(x)$	$f(x)$
$x < -2$	$-$	$-$	$+$	$-$	$-$	$\searrow$
$-2 < x < 0$	$-$	$+$	$-$	$+$	$+$	$\nearrow$
$x > 0$	$+$	$+$	$+$	$-$	$-$	$\searrow$

Second derivative

$$f''(x) = \left(-\frac{x^2+2x}{x^2+x+1}\right)' =$$

$$-\frac{(2x+2)(x^2+x+1)^2-(x^2+2x)\left((x^2+x+1)^2\right)'}{(x^2+x+1)^4} =$$

$$-\frac{(2x+2)(x^2+x+1)^2-(x^2+2x)2(2x+1)(x^2+x+1)}{(x^2+x+1)^4} =$$

$$-\frac{(2x+2)(x^2+x+1)-2(x^2+2x)(2x+1)}{(x^2+x+1)^3} =$$

$$-\frac{2x^3+2x^2+2x^2+2x+2x+2-4x^3-8x^2-2x^2-4x}{(x^2+x+1)^3} =$$

$$-\frac{-2x^3-6x^2+2}{(x^2+x+1)^3} = 2\frac{x^3+3x^2-2}{(x^2+x+1)^3}$$

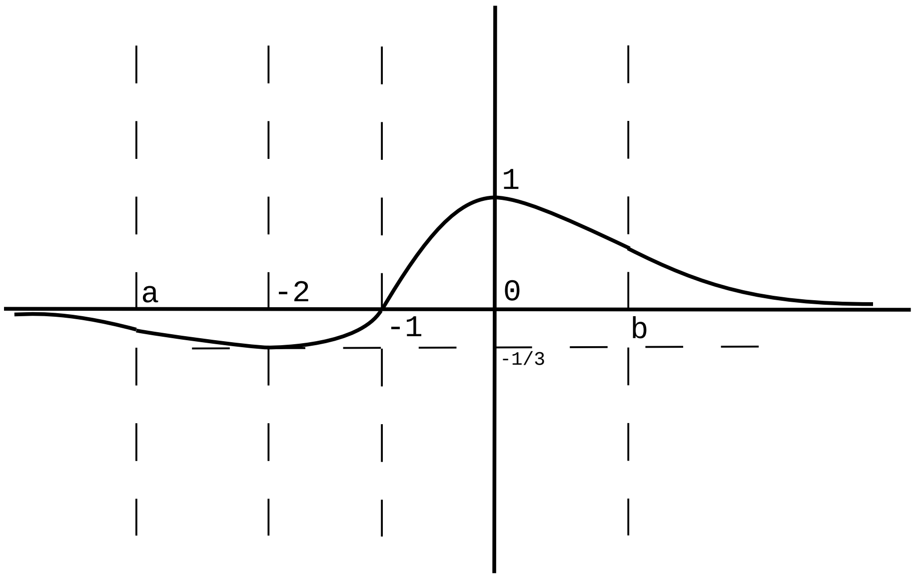
$x=-1$ is obviously a root of x^3+3x^2-2:

$$x^3+3x^2-2 = (x+1)(x^2+2x-2) = (x+1)(x+1+\sqrt{5})(x+1-\sqrt{5})$$

Denominator is always positive so the sign of $f''(x)$ depends only on the sign of $(x+1+\sqrt{5})(x+1)(x+1-\sqrt{5})$, roots are arranged in the following order $-1-\sqrt{5} < -1 < \sqrt{5}-1$. Let's find sign of $f''(x)$:

	$x+1+\sqrt{5}$	$x+1$	$x+1-\sqrt{5}$	x^3+3x^2-2	$f''(x)$	$f(x)$
$x < -1-\sqrt{5}$	$-$	$-$	$-$	$-$	$-$	$\frown$
$-1-\sqrt{5} < x < -1$	$+$	$-$	$-$	$+$	$+$	$\smile$
$-1 < x < \sqrt{5}-1$	$+$	$+$	$-$	$-$	$-$	$\frown$
$x > \sqrt{5}-1$	$+$	$+$	$+$	$+$	$+$	$\smile$

Sketch



where $a=-1-\sqrt{5}$, $b=\sqrt{5}-1$.

10. (a)

$$\int_0^\pi \sin(2-x)\,dx = (*)$$

Let's say $u = 2 - x$, then $du = -dx$, $u(0) = 2$, $u(\pi) = 2 - \pi$ and:

$$(*) = -\int_2^{2-\pi} \sin u \, du = -(-\cos u)\Big|_2^{2-\pi} = \cos u\Big|_2^{2-\pi}$$

$$= \cos(2-\pi) - \cos 2$$

(b)

$$\int \frac{\sqrt[3]{\frac{1}{x^2}+1}}{x^3}\, dx = (*)$$

Let $u = \frac{1}{x^2} + 1 = x^{-2} + 1$, then

$$du = (-2)x^{-3}dx$$

$$-\frac{du}{2} = \frac{dx}{x^3}$$

$$(*) = \int \sqrt[3]{u}\left(-\frac{du}{2}\right) = -\frac{1}{2}\int \sqrt[3]{u}\, du = -\frac{1}{2}\int u^{\frac{1}{3}}\, du =$$

$$-\frac{1}{2}\frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C = -\frac{3u^{\frac{4}{3}}}{8} + C = -\frac{3(\frac{1}{x^2}+1)^{\frac{4}{3}}}{8} + C$$

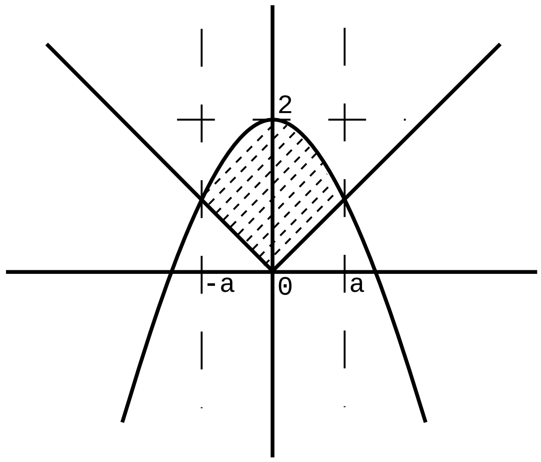
(c)

$$\int \frac{(1+\tan x)^3}{\cos^2 x}\, dx = (*)$$

Let $u = 1 + \tan x$, then $du = \frac{dx}{\cos^2 x}$ and:

$$(*) = \int u^3\, du = \frac{u^4}{4} + C = \frac{(1+\tan x)^4}{4} + C$$

11. Here is the graph:



The area we have to find is symmetrical about y-axis (because both functions are even), hence

$$A = 2\int_0^a (2-x) - |x|\,dx$$

$|x| = x$ for $x > 0$ and a is such that

$$2 - a = |a|$$

$$2 - a = a$$

$$2 = 2a$$

$$a = 1$$

No we are ready to find the area between the curves:

$$A = 2\int_0^1 (2-x) - x\,dx = 2\int_0^1 2 - 2x\,dx = 4\int_0^1 1 - x\,dx =$$

$$4(x - \frac{x^2}{2})\Big|_0^1 = 4(1 - \frac{1}{2} - 0 + 0) = 4 - 2 = 2$$

12.

$$s(t) = \int_0^t 1 + \frac{1}{\sqrt{x}}\,dx = \int_0^t 1 + x^{-\frac{1}{2}}\,dx = x + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\Big|_0^t = x + 2x^{\frac{1}{2}}\Big|_0^t = t + 2\sqrt{t}$$